CHARACTERIZATION OF RANK TWO LOCALLY NILPOTENT DERIVATIONS IN DIMENSION THREE

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Abstract. In this paper we give an algorithmic characterization of rank two locally nilpotent derivations in dimension three. Together with an algorithm for computing the plinth ideal, this gives a method for computing the rank of a locally nilpotent derivation in dimension three.

1. Introduction

Let $\mathcal{K}$ be a commutative field of characteristic zero, $\mathcal{K}^{[n]}$ be the ring of polynomials in n variables with coefficients in $\mathcal{K}$ and $Aut_{\mathcal{K}}(\mathcal{K}^{[n]})$ be the group of $\mathcal{K}$-automorphisms of $\mathcal{K}^{[n]}$. The structure of $Aut_{\mathcal{K}}(\mathcal{K}^{[2]})$ is well understood [21]. However, $Aut_{\mathcal{K}}(\mathcal{K}^{[n]})$ remains a big mystery for $n \geq 3$.

In order to understand the nature of $Aut_{\mathcal{K}}(\mathcal{K}^{[n]})$ it is natural to investigate algebraic group actions on the affine n-space over $\mathcal{K}$. Actions of the algebraic group $(\mathcal{K}, +)$ on affine spaces are commonly called *algebraic G_a-actions*, and they are all of the form $\exp(t\mathcal{X})_{t\in\mathcal{K}}$ where $\mathcal{X}$ is a locally nilpotent $\mathcal{K}$-derivation of the polynomial ring $\mathcal{K}^{[n]}$.

Locally nilpotent derivations of $\mathcal{K}^{[2]}$ are completely classified, and this classification is algorithmic [25]. In dimension three, D. Daigle, G. Freudenburg and S. Kaliman obtained several deep results which constitute a big step towards a classification, see [17] and the references therein. However, some of these results which are obtained by using topological methods are not of algorithmic nature. It would of course be very nice to obtain an algorithmic classification of locally nilpotent derivations in dimension three, but this seems to be a difficult problem. This paper addresses the less ambitious problem of computing some invariants, namely the plinth ideal and the rank, of a locally nilpotent derivation in dimension three.

The paper is structured as follows. In section 2 we set up notation and give the main results to be used. Section 3 concerns minimal local slices and how to compute them in dimension 3. This gives an algorithm for computing a generator of the plinth ideal of a locally nilpotent derivation in dimension three. In section 4 we give an algorithm to compute the rank of a locally nilpotent $\mathcal{K}$-derivation in dimension 3.

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2. Notation and basic facts

Throughout this paper $\mathcal{K}$ is a commutative field of characteristic zero, all the considered rings are commutative of characteristic zero with unit and all the considered derivations are nonzero. A derivation of a $\mathcal{K}$-algebra $\mathcal{A}$ is called a $\mathcal{K}$-derivation if it satisfies $\mathcal{X}(a) = 0$ for any $a \in \mathcal{K}$.

2.1. **Classical results on locally nilpotent derivations.** A derivation $\mathcal{X}$ of a ring $\mathcal{A}$ is called *locally nilpotent* if for any $a \in \mathcal{A}$ there exists a positive integer n such that $\mathcal{X}^n(a) = 0$. An element s of $\mathcal{A}$ satisfying $\mathcal{X}(s) \neq 0$ and $\mathcal{X}^2(s) = 0$ is called a *local slice* of $\mathcal{X}$. If moreover $\mathcal{X}(s) = 1$ then s is called a *slice* of $\mathcal{X}$. A nonzero locally nilpotent derivation needs not to have a slice but always has a local slice.

The following result, which dates back at least to [27], concerns locally nilpotent derivations having a slice.

Lemma 2.1. *Let $\mathcal{A}$ be a ring containing $\mathbb{Q}$ and $\mathcal{X}$ be a locally nilpotent derivation of $\mathcal{A}$ having a slice s. Then $\mathcal{A} = \mathcal{A}^{\mathcal{X}}[s]$ and $\mathcal{X} = \partial_s$.*

Let $\mathcal{A}$ be a ring and $\mathcal{X}$ be a derivation of $\mathcal{A}$. The subset $\{a \in \mathcal{A};\ \mathcal{X}(a) = 0\}$ of $\mathcal{A}$ is in fact a subring called the *ring of constants* of $\mathcal{X}$ and is denoted by $\mathcal{A}^{\mathcal{X}}$. When $\mathcal{A}$ is a domain and $\mathcal{X}$ is locally nilpotent, the ring of constants $\mathcal{A}^{\mathcal{X}}$ is factorially closed in $\mathcal{A}$, i.e., if $a \in \mathcal{A}^{\mathcal{X}}$ and $a = bc$ then $b, c \in \mathcal{A}^{\mathcal{X}}$. Also, the fact that $\mathcal{A}$ is of characteristic zero implies that $\mathcal{A}^{\star} \subset \mathcal{A}^{\mathcal{X}}$.

Locally nilpotent derivations in two variables over fields of characteristic 0 are well understood. We have in particular the following version of Rentschler's theorem [25].

Theorem 2.2. *Let $\mathcal{X}$ be a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{K}[x, y]$. Then there exist two polynomials f, g of $\mathcal{K}[x, y]$ and a univariate polynomial h such that $\mathcal{K}[f, g] = \mathcal{K}[x, y]$, $\mathcal{K}[x, y]^{\mathcal{X}} = \mathcal{K}[f]$ and $\mathcal{X} = h(f)\partial_g$.*

As a consequence of theorem 2.2, if $\mathcal{A}$ is a UFD containing $\mathbb{Q}$ and $\mathcal{X}$ is a locally nilpotent $\mathcal{A}$-derivation of $\mathcal{A}[x, y]$ then there exists $f \in \mathcal{A}[x, y]$ and a univariate polynomial h such that $\mathcal{A}[x, y]^{\mathcal{X}} = \mathcal{A}[f]$ and $\mathcal{X} = h(f)(\partial_y f \partial_x - \partial_x f \partial_y)$, see [9, 12].

In the case of $\mathcal{K}^{[3]}$ we have the following result proved by Miyanishi [24] for the case $\mathcal{K} = \mathbb{C}$ and may be extended to the general case in a straightforward way by using Kambayashi's transfer principle [20], see also [6] for an algebraic proof.

Theorem 2.3. *Let $\mathcal{X}$ be a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{K}[x, y, z]$. Then there exist polynomials f, g such that $\mathcal{K}[x, y, z]^{\mathcal{X}} = \mathcal{K}[f, g]$.*

Contrary to lemma 2.1 and theorem 2.2 which are of algorithmic nature, it is not clear from the existing proofs of theorem 2.3 how to compute, for a given locally nilpotent derivation $\mathcal{X}$ of $\mathcal{K}[x, y, z]$, two polynomials f, g such that $\mathcal{K}[x, y, z]^{\mathcal{X}} = \mathcal{K}[f, g]$.

2.2. **Coordinates.** A polynomial $f \in \mathcal{K}[x_1, \ldots, x_n]$ is called a *coordinate* if there exists a list of polynomials $f_1, \ldots, f_{n-1}$ such that $\mathcal{K}[x_1, \ldots, x_n] = \mathcal{K}[f, f_1, \ldots, f_{n-1}]$. In the same way a list $f_1, \ldots, f_r$ of polynomials, with $r \leq n$, is called a *system of coordinates* if there exists a list $f_{r+1}, \ldots, f_n$ of polynomials such that $\mathcal{K}[x_1, \ldots, x_n] = \mathcal{K}[f_1, \ldots, f_n]$. A system of coordinates of length n is called a *coordinate system* of $\mathcal{K}[x_1, \ldots, x_n]$.

The famous Abhyankar-Moh theorem [1] states that a polynomial f in $\mathcal{K}[x,y]$ is a coordinate if and only if $\mathcal{K}[x,y]/f$ is $\mathcal{K}$-isomorphic to $\mathcal{K}^{[1]}$. In the case of three variables we have the following result proved by Kaliman in [19] for the case $\mathcal{K}=\mathbb{C}$ and extended to the general case in [10] by using Kambayashi's transfer principle [20].

Theorem 2.4. *Let f be a polynomial in $\mathcal{K}[x,y,z]$ and assume that for all but finitely many $\alpha \in \mathcal{K}$ the $\mathcal{K}$-algebra $\mathcal{K}[x,y,z]/(f-\alpha)$ is $\mathcal{K}$-isomorphic to $\mathcal{K}^{[2]}$. Then f is a coordinate of $\mathcal{K}[x,y,z]$.*

As a consequence of theorem 2.4, if a polynomial f is such that $\mathcal{K}(f)[x,y,z]$ is $\mathcal{K}(f)$-isomorphic to $\mathcal{K}(f)^{[2]}$ then f is a coordinate [23, 13]. This is the version of Kaliman's theorem we will use in this paper. However, it is not clear how to compute polynomials g,h such that f,g,h is a coordinate system of $\mathcal{K}[x,y,z]$ since the original proof given in [19] is of topological nature.

2.3. **Rank of a derivation.** Let $\mathcal{X}$ be a $\mathcal{K}$-derivation of $\mathcal{K}[\underline{x}]=\mathcal{K}[x_1,\ldots,x_n]$. As defined in [16] the co-rank of $\mathcal{X}$, denoted by $corank(\mathcal{X})$, is the unique nonnegative integer r such that $\mathcal{K}[\underline{x}]^{\mathcal{X}}$ contains a system of coordinates of length r and no system of coordinates of length greater than r. The rank of $\mathcal{X}$, denoted by $rank(\mathcal{X})$, is defined by $rank(\mathcal{X}) = n - corank(\mathcal{X})$. Intuitively, the rank of $\mathcal{X}$ is the minimal number of partial derivatives needed to express $\mathcal{X}$. The only one derivation of rank 0 is the zero derivation. Any $\mathcal{K}$-derivation of rank 1 is of the form $p(f_1,\ldots,f_{n-1})\partial_{f_n}$, where $f_1,\ldots,f_n$ is a coordinate system. Such a derivation is locally nilpotent if and only if p does not depend on f_n.

Let $\mathcal{X}$ be a nonzero locally nilpotent $\mathcal{K}$-derivation of $\mathcal{K}[x_1,\ldots,x_n]$, write $\mathcal{X} = \sum a_i\partial_{x_i}$ and set $c=\gcd(a_1,\ldots,a_n)$. We say that $\mathcal{X}$ is irreducible if c is a constant of $\mathcal{K}^\star$. It is well known that $\mathcal{X}(c)=0$ and $\mathcal{X}=c\mathcal{Y}$ where $\mathcal{Y}$ is an irreducible locally nilpotent $\mathcal{K}$-derivation. Moreover, this decomposition is unique up to a unit, i.e., if $\mathcal{X}=c_1\mathcal{Y}_1$, where $\mathcal{Y}_1$ is irreducible, then there exists an nonzero constant $\mu \in \mathcal{K}^\star$ such that $c_1=\mu c$ and $\mathcal{Y}=\mu\mathcal{Y}_1$. Given any irreducible locally nilpotent $\mathcal{K}$-derivation of $\mathcal{K}[x_1,\ldots,x_n]$ and any c such that $\mathcal{X}(c)=0$, the derivations $\mathcal{X}$ and $c\mathcal{X}$ have the same rank. Thus, for rank computation we may reduce, without loss of generality, to irreducible derivations.

2.4. **The plinth ideal of a locally nilpotent derivation.** Let $\mathcal{A}$ be a ring, $\mathcal{X}$ be a locally nilpotent derivation of $\mathcal{A}$ and let

$$\mathcal{S}^{\mathcal{X}} := \{\mathcal{X}(a) \mid \mathcal{X}^2(a)=0\}.$$

Clearly $\mathcal{S}^{\mathcal{X}}$ is an ideal of $\mathcal{A}^{\mathcal{X}}$, called the *plinth ideal* of $\mathcal{X}$. In case $\mathcal{A}=\mathcal{K}[x,y,z]$ we have the following result from [10] which is a consequence of faithful flatness of $\mathcal{K}[x,y,z]$ as $\mathcal{K}[x,y,z]^{\mathcal{X}}$-module.

Theorem 2.5. *Let $\mathcal{X}$ be a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{K}[x,y,z]$. Then the ideal $\mathcal{S}^{\mathcal{X}}$ is principal.*

As we will see in the sequel, the ideal $\mathcal{S}^{\mathcal{X}}$ contains a crucial information for computing the rank of a locally nilpotent derivation in dimension 3. Before going on to the details on how to exploit this information for our purpose we first focus on how to compute a generator of this ideal.

3. Minimal local slices

In this section we give an algorithm to compute a generator of the ideal $\mathcal{S}^{\mathcal{X}}$ in case $\mathcal{X}$ is a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{K}[x,y,z]$. Besides theorem 2.5, our algorithm strongly depends on the fact that $\mathcal{K}[x,y,z]^{\mathcal{X}}$ is generated by 2 polynomials. Since we do not have at disposal an algorithmic version of Miyanishi theorem we assume a generating system of $\mathcal{K}[x,y,z]^{\mathcal{X}}$ to be available.

Definition 3.1. *Let $\mathcal{A}$ be a domain and $\mathcal{X}$ be a locally nilpotent derivation of $\mathcal{A}$. A local slice s of $\mathcal{X}$ is called minimal if for any local slice v such that $\mathcal{X}(v) \mid \mathcal{X}(s)$ we have $\mathcal{X}(v) = \mu\mathcal{X}(s)$, where μ is a unit of $\mathcal{A}$.*

Lemma 3.2. *Let $\mathcal{A}$ be a domain and $\mathcal{X}$ be a locally nilpotent derivation of $\mathcal{A}$. Let s be a local slice of $\mathcal{X}$, p be a factor of $\mathcal{X}(s) = c$ and write $c = pc_1$. Then there exists $s_1 \in \mathcal{A}$ such that $\mathcal{X}(s_1) = c_1$ if and only if the ideal $p\mathcal{A}$ contains an element of the form $s + a$ where $a \in \mathcal{A}^{\mathcal{X}}$.*

Proof. "$\Rightarrow$") Assume that there exists a local slice s_1 of $\mathcal{X}$ such that $\mathcal{X}(s_1) = c_1$. Then $\mathcal{X}(ps_1 - s) = 0$ and so $ps_1 - s = a$ where a is a constant of $\mathcal{X}$. This proves that $p\mathcal{A}$ contains $s + a$.

"$\Leftarrow$") Assume now that the ideal $p\mathcal{A}$ contains an element of the form $s + a$, where a is a constant of $\mathcal{X}$, and write $s + a = ps_1$. Then $\mathcal{X}(s) = p\mathcal{X}(s_1)$ and so $\mathcal{X}(s_1) = c_1$. □

Proposition 3.3. *Let $\mathcal{A}$ be a UFD, $\mathcal{X}$ be a locally nilpotent derivation of $\mathcal{A}$ and s be a local slice of $\mathcal{X}$. Then the following hold:*

i) there exists a minimal local slice s_0 of $\mathcal{X}$ such that $\mathcal{X}(s_0) \mid \mathcal{X}(s)$,

ii) in case $\mathcal{S}^{\mathcal{X}}$ is a principal ideal, it is generated by $\mathcal{X}(s)$ for any minimal local slice s of $\mathcal{X}$.

Proof. i) Let s be a local slice of $\mathcal{X}$ and write $\mathcal{X}(s) = \mu p_1^{m_1} \cdots p_r^{m_r}$, where μ is a unit and the p_i's are primes, and set $m = \sum_i m_i$. We will prove the result by induction on m.

For $m = 0$ we have $\mathcal{X}(s) = \mu$, and so $\mu^{-1}s$ is a slice of $\mathcal{X}$. This shows that s is a minimal local slice of $\mathcal{X}$. Let us now assume the result to hold for $m-1$ and let s be a local slice of $\mathcal{X}$, with $\mathcal{X}(s) = \mu p_1^{m_1} \cdots p_r^{m_r}$ and $\sum_i m_i = m$. Then we have one of the following cases.

- For any $i = 1, \ldots, r$ the ideal $p_i\mathcal{A}$ does not contain any element of the form $s + a$ with $\mathcal{X}(a) = 0$. In this case s is a minimal local slice of $\mathcal{X}$ by lemma 3.2.

- There exists i such that $p_i\mathcal{A}$ contains an element of the form $s + a$, with $\mathcal{X}(a) = 0$. Without loss of generality we may assume that $i = 1$. If we write $s + a = p_1 s_1$ then $\mathcal{X}(s_1) = p_1^{m_1-1} p_2^{m_2} \cdots p_r^{m_r}$, and by using induction hypothesis we get a minimal local slice s_0 of $\mathcal{X}$ such that $\mathcal{X}(s_0) \mid \mathcal{X}(s_1)$. Since $\mathcal{X}(s_1) \mid \mathcal{X}(s)$ we get the result in this case.

ii) Assume now that $\mathcal{S}^{\mathcal{X}}$ is principal and let c be a generator of this ideal, with $c = \mathcal{X}(s_0)$. Let s be a minimal local slice of $\mathcal{X}$. Since $\mathcal{X}(s) \in \mathcal{S}^{\mathcal{X}}$ we may write $\mathcal{X}(s) = \mu\mathcal{X}(s_0)$. The fact that s is minimal implies that μ is a unit of $\mathcal{A}^{\mathcal{X}}$, and so $\mathcal{X}(s)$ generates $\mathcal{S}^{\mathcal{X}}$. □

The main question to be addressed, if we want to have an algorithmic version of proposition 3.3, is to check, for a given prime p of $\mathcal{A}$, whether $p\mathcal{A} \cap \mathcal{A}^{\mathcal{X}}[s]$ contains a monic polynomial of degree 1 with respect to s. In case $\mathcal{A}$ is an affine ring over

a computable field $\mathcal{K}$ this problem may be solved by using Gröbner bases theory, see e.g., [2, 3, 5]. We only deal here with the case where $\mathcal{A}$ and $\mathcal{A}^{\mathcal{X}}$ are polynomial rings over a field since this fits our need.

Proposition 3.4. *Let $\mathcal{I}$ be an ideal of $\mathcal{K}[x_1,\dots,x_n] = \mathcal{K}[\underline{x}]$ and $\underline{h} = h_1,\dots,h_t$ be a list of algebraically independent polynomials of $\mathcal{K}[\underline{x}]$. Let $\underline{u} = u_1,\dots,u_t$ be a list of new variables and $\mathcal{J}$ be the ideal of $\mathcal{K}[\underline{u},\underline{x}]$ generated by $\mathcal{I}$ and $h_1-u_1,\dots,h_t-u_t$. Let G be a Gröbner basis of $\mathcal{J}$ with respect to the lexicographic order $u_1 \prec \cdots \prec u_t \prec x_1 \prec \cdots \prec x_n$, and $\{g_1,\dots,g_v\} = G \cap \mathcal{K}[\underline{u}]$. Then:*

i) $\{g_1,\dots,g_v\}$ is a Gröbner basis of $\mathcal{J} \cap \mathcal{K}[\underline{u}]$ with respect to the lexicographic order $u_1 \prec \cdots \prec u_t$,

ii) the $\mathcal{K}$-isomorphism $u_i \in \mathcal{K}[\underline{u}] \longmapsto h_i \in \mathcal{K}[\underline{h}]$ maps $\mathcal{J} \cap \mathcal{K}[\underline{u}]$ onto $\mathcal{I} \cap \mathcal{K}[\underline{h}]$.

In our case, we have $\mathcal{I} = p\mathcal{K}[x,y,z]$ for some polynomial p, and $\mathcal{K}[\underline{h}] = \mathcal{K}[f,g,s]$ where f,g is a generating system of $\mathcal{K}[x,y,z]^{\mathcal{X}}$ and s is a local slice of $\mathcal{X}$. Let u_1,u_2,u_3 be new variables and $\mathcal{J}$ be the ideal of $\mathcal{K}[u_1,u_2,u_3,x,y,z]$ generated by $p, f-u_1, g-u_2, s-u_3$. Let G be a Gröbner basis of $\mathcal{J}$ with respect to the lexicographic order $u_1 \prec u_2 \prec u_3 \prec x \prec y \prec z$ and $G_1 = G \cap \mathcal{K}[u_1,u_2,u_3]$. According to proposition 3.4, the ideal $p\mathcal{K}[x,y,z] \cap \mathcal{K}[f,g,s]$ contains a polynomial of the form $s + a(f,g)$ if and only if G_1 contains a monic polynomial $\ell(u_1,u_2,u_3)$ of degree 1 with respect to u_3. In this case the polynomial we are looking for is $\ell(f,g,s)$.

The following algorithm gives the main steps to be performed for computing a minimal local slice of a given locally nilpotent derivation in dimension 3.

Input: A locally nilpotent $\mathcal{K}$-derivation $\mathcal{X}$ of $\mathcal{K}[x,y,z]$ and a generating system f,g of $\mathcal{K}[x,y,z]^{\mathcal{X}}$.
Output: A minimal local slice s of $\mathcal{X}$.

```
Compute a local slice s_0 of X.
Write X(s_0) = p_1^{m_1} ··· p_r^{m_r}, where the p_i's are primes.
s := s_0.
for i from 1 to r do
   for j from 1 to m_i do
      Let G be a Gröbner basis of I(p_i, f − u_1, g − u_2, s − u_3) with respect to
      the lex-order u_1 ≺ u_2 ≺ u_3 ≺ x ≺ y ≺ z, and let G_1 = G ∩ K[u_1, u_2, u_3].
      if G_1 contains a monic polynomial of degree 1 with respect to u_3, say
      u_3 + a(u_1, u_2) then
         Write s + a(f, g) = p_i s_1.
         s := s_1.
      else
         Break.
      end if
   end for
end for
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Algorithm 1: : Minimal local slice algorithm.

4. Computing the rank in dimension three

An irreducible locally nilpotent $\mathcal{K}$-derivation $\mathcal{X}$ of $\mathcal{K}[x_1,\dots,x_n]$ is of rank 1 if and only if $\mathcal{K}[x_1,\dots,x_n]^{\mathcal{X}} = \mathcal{K}^{[n-1]}$ and $\mathcal{X}$ has a slice, see [16]. In dimension 3, and taking into account theorem 2.3, an irreducible locally nilpotent derivation is of rank one if and only if Algorithm 1 produces a slice. Therefore, we only need to characterize derivations of rank two.

Theorem 4.1. *Let $\mathcal{X}$ be an irreducible locally nilpotent derivation of $\mathcal{K}[x,y,z]$ and assume that $rank(\mathcal{X}) \neq 1$. Let us write $\mathcal{K}[x,y,z]^{\mathcal{X}} = \mathcal{K}[f,g]$ and $\mathcal{S}^{\mathcal{X}} = c\mathcal{K}[f,g]$. Then the following are equivalent:*

i) $rank(\mathcal{X}) = 2$,

ii) $c = \ell(u)$, where ℓ is a univariate polynomial and u is a coordinate of $\mathcal{K}[f,g]$,

iii) $c = \ell(u)$, where u is a coordinate of $\mathcal{K}[x,y,z]$.

Proof. $i) \Rightarrow ii)$ Assume that $rank(\mathcal{X}) = 2$ and let u, v, w be a coordinate system such that $\mathcal{X}(u) = 0$. The $\mathcal{K}$-derivation $\mathcal{X}$ is therefore a $\mathcal{K}[u]$-derivation of $\mathcal{K}[u][v,w]$, and since $\mathcal{K}[u]$ is a UFD there exists $p \in \mathcal{K}[x,y,z]$ such that $\mathcal{K}[f,g] = \mathcal{K}[u,p]$. This proves that u is a coordinate of $\mathcal{K}[f,g]$. Let us now view $\mathcal{X}$ as $\mathcal{K}(u)$-derivation of $\mathcal{K}(u)[v,w]$. Since $\mathcal{X}$ is irreducible and according to theorem 2.2 there exists $s = \frac{h(u,v,w)}{k(u)}$ such that $\mathcal{X}(s) = 1$, and so $\mathcal{X}(h) = k(u)$. Let c be a generator of $\mathcal{S}^{\mathcal{X}}$. Then $c \mid k(u)$, and since $\mathcal{K}[u]$ is factorially closed in $\mathcal{K}[u,v,w]$ we have $c = \ell(u)$ for some univariate polynomial ℓ.

$ii) \Rightarrow iii)$ Assume that $c = \ell(u)$, where u is a coordinate of $\mathcal{K}[f,g]$ and write $\mathcal{K}[f,g] = \mathcal{K}[u,p]$. Let s be such that $\mathcal{X}(s) = c$. If we view $\mathcal{X}$ as $\mathcal{K}(u)$-derivation of $\mathcal{K}(u)[x,y,z]$ then $\mathcal{K}(u)[x,y,z]^{\mathcal{X}} = \mathcal{K}(u)[p]$ and $\mathcal{X}(c^{-1}s) = 1$. By applying lemma 2.1 we get $\mathcal{K}(u)[x,y,z] = \mathcal{K}(u)[p,s]$. From theorem 2.4 we deduce that u is a coordinate of $\mathcal{K}[x,y,z]$.

$iii) \Rightarrow i)$ We have $\mathcal{X}(c) = \ell'(u)\mathcal{X}(u) = 0$, and so $\mathcal{X}(u) = 0$. On the other hand, since u is assumed to be a coordinate of $\mathcal{K}[x,y,z]$ we have $rank(\mathcal{X}) \leq 2$. By assumption we have $rank(\mathcal{X}) \neq 1$ and so $rank(\mathcal{X}) = 2$. □

The conditions of $ii)$ in theorem 4.1 are in fact algorithmic. Indeed, it is algorithmically possible to check whether a given polynomial in two variables is a coordinate, see e.g. [1, 4, 11, 26]. We will use the algorithm given in [11], but it is worth mentioning that from the complexity point of view the algorithm given in [26] is the most efficient as reported in [22]. On the other hand, condition $c = \ell(u)$ may be checked by using a special case, called *uni-multivariate decomposition*, of functional decomposition of polynomials, see e.g., [18]. It is important to notice here that uni-multivariate decomposition is essentially unique. Namely, if $c = \ell(u) = \ell_1(u_1)$, where u and u_1, are undecomposable, then there exist $\mu \in \mathcal{K}^\star$ and $\nu \in \mathcal{K}$ such that $u_1 = \mu u + \nu$. Due to the particular nature of our decomposition problem it seems more convenient to use the following proposition.

Proposition 4.2. *Let $c(\underline{x}) \in \mathcal{K}[x_1,\dots,x_n]$ be nonconstant and $\underline{u} = u_1,\dots,u_n$ be a list of new variables. Then the following are equivalent:*

i) $c(\underline{x}) = \ell(y_1(\underline{x}))$, where ℓ is a univariate polynomial and y_1 is a coordinate of $\mathcal{K}[x_1,\dots,x_n]$,

ii) $y_1(\underline{x}) - y_1(\underline{u}) \mid c(\underline{x}) - c(\underline{u})$ and y_1 is a coordinate of $\mathcal{K}[x_1,\dots,x_n]$.

Proof. $i) \Rightarrow ii)$ Let ℓ be a univariate polynomial and t, t' be new variables. Then we have $t - t' \mid \ell(t) - \ell(t')$. This shows that $y_1(\underline{x}) - y_1(\underline{u}) \mid \ell(y(\underline{x})) - \ell(y(\underline{u}))$.

ii) $\Rightarrow$ *i)* Let $y_2, \ldots, y_n$ be polynomials such that $\underline{y} = y_1, \ldots, y_n$ is a coordinate system of $\mathcal{K}[\underline{x}]$, and let $v_i = y_i(\underline{u})$. Then $\underline{v} = v_1, \ldots, v_n$ is a coordinate system of $\mathcal{K}[\underline{u}]$. Let us write $c(\underline{x}) - c(\underline{u}) = (y_1(\underline{x}) - y_1(\underline{u}))A(\underline{u}, \underline{x})$ and $c(\underline{x}) = \ell(\underline{y})$. Then we have

$$\ell(\underline{y}) - \ell(\underline{v}) = (y_1 - v_1)B(\underline{v}, \underline{y}). \tag{4.1}$$

Let us now write $\ell(\underline{y}) = \sum_\alpha a_\alpha(y_1) y_2^{\alpha_2} \cdots y_n^{\alpha_n}$, where $\alpha = (\alpha_2, \ldots, \alpha_n)$. After substituting y_1 to v_1 in the relation (4.1) we get

$$\sum_\alpha a_\alpha(y_1) y_2^{\alpha_2} \cdots y_n^{\alpha_n} - \sum_\alpha a_\alpha(y_1) v_2^{\alpha_2} \cdots v_n^{\alpha_n} = 0.$$

Taking into account the fact that $v_2, \ldots, v_n, y_2, \ldots, y_n$ are algebraically independent over $\mathcal{K}[y_1]$ we get $a_\alpha = 0$ for any $\alpha \neq 0$. This proves that $\ell(\underline{y})$ is a polynomial in terms of y_1. □

Input: A locally nilpotent $\mathcal{K}$-derivation $\mathcal{X}$ of $\mathcal{K}[x, y, z]$ and a generating system f, g of $\mathcal{K}[x, y, z]^{\mathcal{X}}$.
Output: The rank of $\mathcal{X}$.

Write $\mathcal{X} = a_1\partial_x + a_2\partial_y + a_3\partial_z$. Let $c_1 = \gcd(a_1, a_2, a_3)$ and $\mathcal{X} = c_1\mathcal{Y}$.
By Algorithm 1, compute a minimal local slice s of $\mathcal{Y}$ and let $c = \mathcal{Y}(s)$.
if c is a unit **then**
  $rank(\mathcal{X}) = 1$.
else
  Compute a factorization of $c(f, g) - c(t_1, t_2)$ in $\mathcal{K}[f, g, t_1, t_2]$.
  if no factor of $c(f, g) - c(t_1, t_2)$ is of the form $u(f, g) - u(t_1, t_2)$ **then**
    $rank(\mathcal{X}) = 3$.
  else
    if u is a coordinate of $\mathcal{K}[f, g]$ for a factor of the form $u(f, g) - u(t_1, t_2)$ of $c(f, g) - c(t_1, t_2)$ **then**
      $rank(\mathcal{X}) = 2$.
    else
      $rank(\mathcal{X}) = 3$.
    end if
  end if
end if

Algorithm 2: : Rank algorithm.

As defined in [7], a derivation $\mathcal{X}$ of rank r of $\mathcal{K}[\underline{x}]$ is rigid if for any coordinate systems $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ such that $\mathcal{K}[y_1, \ldots, y_{n-r}] \subseteq \mathcal{K}[\underline{x}]^{\mathcal{X}}$ and $\mathcal{K}[z_1, \ldots, z_{n-r}] \subseteq \mathcal{K}[\underline{x}]^{\mathcal{X}}$ we have $\mathcal{K}[y_1, \ldots, y_{n-r}] = \mathcal{K}[z_1, \ldots, z_{n-r}]$. The main results of [7] lie behind the fact that locally nilpotent derivations in dimension 3 are rigid. In fact only the rank two case is nontrivial since in general derivations of rank $0, 1$ and n are obviously rigid. The characterization *ii)* of rank two derivations given in theorem 4.1 gives a more precise information. Indeed, it tells that if a coordinate of $\mathcal{K}[x, y, z]$ belongs to $\mathcal{K}[x, y, z]^{\mathcal{X}}$ then it may be found by decomposing the generator of the plinth ideal $\mathcal{S}^{\mathcal{X}}$. The fact that rank two derivations are

rigid is then an obvious consequence of the uniqueness property of uni-multivariate decomposition.

In case a given derivation $\mathcal{X}$ is of rank 2, Algorithm 2 does not produce a coordinate system u, v, w such that $\mathcal{X}(u) = 0$. Computing such a coordinate system, which is equivalent to obtaining an algorithmic version of theorem 2.4, is a necessary step towards our ultimate goal, namely algorithmically classifying locally nilpotent derivations in dimension 3.

5. Comments on implementation

Before implementing algorithms for locally nilpotent derivations of $\mathcal{K}[x, y, z]$ we must first specify how such objects are to be concretely represented. Any chosen representation should address the two following problems.

(1) *Recognition problem:* Given a $\mathcal{K}$-derivation $\mathcal{X}$ of $\mathcal{K}[x, y, z]$, check whether $\mathcal{X}$ is locally nilpotent.
(2) *Kernel problem:* Given a locally nilpotent $\mathcal{K}$-derivation $\mathcal{X}$ of $\mathcal{K}[x, y, z]$, compute f, g such that $\mathcal{K}[x, y, z]^{\mathcal{X}} = \mathcal{K}[f, g]$.

Usually, a derivation $\mathcal{X}$ of $\mathcal{K}[x, y, z]$ is written as a $\mathcal{K}[x, y, z]$-linear combination of the partial derivatives $\partial_x, \partial_y, \partial_z$. However, with such a representation, the recognition and kernel problems are nowhere near completely solved. To our knowledge, only the weighted homogeneous case of the recognition problem is solved, see [15]. One way to go round this hurdle is to opt for another representation.

The Jacobian representation gives another alternative to represent locally nilpotent derivations. Indeed, any irreducible locally nilpotent $\mathcal{K}$-derivation of $\mathcal{K}[x, y, z]$ is, up to a nonzero constant in $\mathcal{K}$, equal to $\mathrm{Jac}(f, g, .)$, see [8]. In order to check whether a Jacobian derivation $\mathcal{X} = \mathrm{Jac}(f, g, .)$ is locally nilpotent it suffices to check that $\mathcal{X}^{d+1}(x) = \mathcal{X}^{d+1}(y) = \mathcal{X}^{d+1}(z) = 0$, where $d = \deg(f)\deg(g)$, see [14]. However, it is still not clear how such a representation could help in solving the kernel problem. Nevertheless, we may always check whether this ring of constants is generated over $\mathcal{K}$ by f, g by using van den Essen's kernel algorithm [14].

Due to the above discussed issues, we have restricted our implementation to the case of derivations of $\mathcal{K}[x, y, z]$ represented in a Jacobian form, say $\mathrm{Jac}(f, g, .)$, and whose ring of constants is generated by f, g. The computer Algebra system we used for implementation is Maple 9.

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